Matroids and Hyperplane Arrangements

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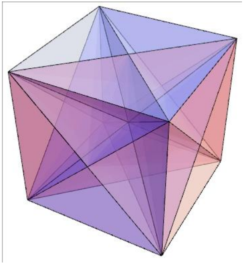
A **hyperplane arrangement** $\mathcal{A} = \{H_1, \ldots H_n\}$ *is a finite set of hyperplanes in V.*

The D_3 arrangement

Example

$$\alpha_1(x,y,z) = x + z$$
$$\alpha_2(x,y,z) = x - z$$
$$\alpha_3(x,y,z) = y + z$$
$$\alpha_4(x,y,z) = y - z$$
$$\alpha_5(x,y,z) = x + y$$
$$\alpha_6(x,y,z) = x - y$$



The arrangement $\mathcal{A} = \{H_1, \ldots, H_6\}$ *in* $\mathbb{R}^3$ *given by the hyperplanes* $H_i = \{(x,y,z) \mid \alpha_i(x,y,z) = 0\}$. *This is the* D_3 *arrangement.*

Why study hyperplane arrangements?

- Ordered configuration space:
  $\widetilde{\mathcal{C}}(\ell, \mathbb{R}^2) = \{(z_1, \dots, z_\ell) \in (\mathbb{R}^2)^\ell \mid z_i \neq z_j, \forall i \neq j\}$

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- The configuration space is the space of possible positions of ℓ distinct particles in $\mathbb{R}^2$.

Example

From the D_3 arrangement, $\{H_2, H_4, H_6\}$ is a minimal dependent set of hyperplanes.

$$\alpha_2(x,y,z) = x - z$$
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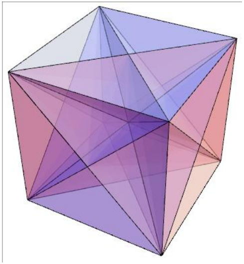
$$\alpha_6(x,y,z) = x - y$$

$$\begin{matrix} \alpha_2 \\ \alpha_4 \\ \alpha_6 \end{matrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$



Definition of a Matroid

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A **matroid** $\mathcal{M}$ *on* E *is an ordered pair* $(E, \mathcal{C})$ *where* E *(called the ground set of* $\mathcal{M}$*) is a finite set and* $\mathcal{C}$ *is a set of subsets (called circuits) of* E *such that*

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A subset of E is defined to be dependent if and only if it contains a circuit.

Matroids of Arrangements

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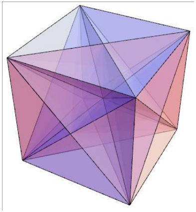

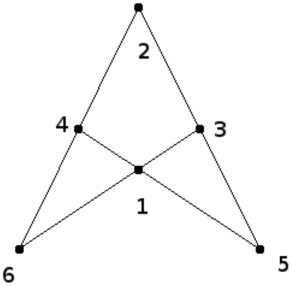

Figure: the matroid $\mathcal{M}$

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Example

$\alpha_2(x, y, z) = x - z$

$$e_2 = \frac{d(x-z)}{x-z} = \frac{dx}{x-z} - \frac{dz}{x-z}$$

The Orlik-Solomon Algebra

- $A = A^0 \oplus A^1 \oplus \cdots \oplus A^n$

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Theorem For any dependent subset of $\mathcal{A}, \{e_{i_1}, \ldots, e_{i_p}\}$,
$\sum_{k=1}^{p} (-1)^{k-1}(e_{i_1} \ldots \hat{e}_{i_k} \ldots e_{i_p}) = 0$ in A, where the $\hat{e}_{i_k}$ element is omitted from the product.

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- The OS algebra $A(\mathcal{A})$ is the cohomology algebra of the complement $\mathbb{C}^\ell - \bigcup_{H \in \mathcal{A}} H$ of the arrangement $\mathcal{A}$, a topological invariant of the complement of $\mathcal{A}$.

Resonance Varieties

- Definition

  The **degree-one resonance variety** $\mathcal{R}^1(\mathcal{A}) = \{a \in A^1 \mid \exists b \in A^1$ *where* $ab = 0$ *and* b *is not a scalar multiple of* $a\}$.

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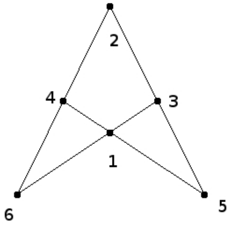
  - $\mathcal{R}^1(\mathcal{A})$ is the union of linear subspaces.
  - The resonance variety is an invariant of the OS algebra, and the degree-1 component of resonance is an invariant of the fundamental group of the complement of the arrangement.

Example

Again, the D_3 arrangement illustrates this structure.



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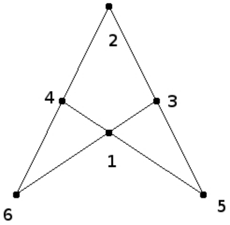
- *The degree-one resonance variety of the D_3 arrangement is the union of five 2-dimensional linear subspaces of $\mathbb{C}$.*



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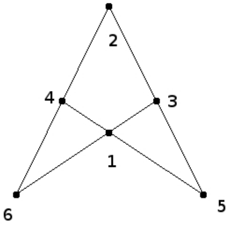
- *The 3-point circuits are the lines in* $\mathcal{M}$: $\{1, 3, 6\}, \{1, 4, 5\}, \{2, 3, 5\}$, *and* $\{2, 4, 6\}$.



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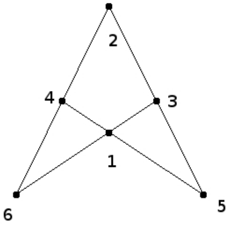
- *The 3-point circuits are the lines in $\mathcal{M}: \{1,3,6\}, \{1,4,5\}, \{2,3,5\}$, and $\{2,4,6\}$.*
- *Each of these yields a 2-dimensional component of $\mathcal{R}^1(\mathcal{M})$.*
  e.g. for 136 the subspace is spanned by $e_1 - e_3$ and $e_3 - e_6$.
  $(e_1 - e_3)(e_3 - e_6) = e_{13} - e_{16} + e_{36} = 0.$



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Sources

- *Arrangements of Hyperplanes* by Peter Orlik and Hiroaki Terao
- *Matroid Theory* by James Oxley
- *Determining Resonance Varieties of Hyperplane Arrangements* by Andres Perez
- The brain of Dr. Michael Falk.